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OPTIMAL COMPLIANCE CRITERION FOR AXISYMMETRIC SOLID PLATES

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Abstract—The problem of minimum compliance of solid plates is formulated in statical terms. It is shown that a previously derived optimality condition is merely a stationary condition. Additional necessary conditions for optimality that distinguish local minima from local maxima are derived from the second variation on the compliance. Although designs which are local minima may exist, it is shown that an absolute minimum does not. An example is presented for which both a local minimum and a local maximum are obtained.

1. INTRODUCTION

This paper treats the problem of minimal compliance design for thin elastic solid plates subject to bending. The design variable is the thickness h, total volume is prescribed, and the compliance is defined as the work done by the external loads. Since the compliance is the negative of the potential energy, minimal compliance designs are maximum potential energy designs or maximum stiffness designs[1, 2]. For special loading conditions, the design for minimum compliance admits alternative physical interpretations. If the loading is a concentrated force or a uniform ring load applied to a circular plate, the compliance is proportional to the vertical displacement at the point or circle of application of the load. Therefore, in this case, the designs for minimal compliance and minimal elastic deflection at the point of application of the load are identical. Similarly, if the applied loading consists of uniform pressure, the design for minimum compliance is equivalent to the design for minimum average deflection.

Basing arguments on the principle of minimum potential energy, previous investigators[1–3] have obtained necessary and sufficient conditions for the maximum stiffness design of sandwich structures. These arguments, however, only lead to an extremal condition for the compliance when solid structures are considered[2]. Since structural arguments alone can not distinguish a local minimum from a saddle point or local maximum, it is necessay to resort to the methods of the calculus of variations. The purpose of this paper is to further explore the criterion for minimal compliance from the variational calculus point of view.

2. FORMULATION OF THE PROBLEM

Consider a thin solid circular plate whose thickness h is a function of the radial coordinate r. The kinematic support conditions, applied pressure $p(r)$ and ring loads P_i acting over the circle $r = r_i$ are prescribed. It is desired to determine the thickness distribution, subject to the condition of prescribed volume, so that the work W done by the applied loads is minimized. If $w(r)$ is the deflection of the middle surface of the plate and w_i is the deflection at $r = r_i$, W becomes

$$W = \int p(r)w(r)\,\mathrm{d}A + 2\pi \sum_i P_i w_i r_i. \tag{1}$$

In terms of the specific stiffness S

$$S^3 = Eh^3/12 \tag{2}$$

the linear elastic response condition is

$$K_r = S^{-3}(M_r - \nu M_\theta) \qquad K_\theta = S^{-3}(M_\theta - \nu M_r) \tag{3}$$

where K_r, K_θ, M_r and M_θ are the radial and circumferential curvatures of the deflected middle surface of the plate and bending moments, respectively, ν is Poisson's ratio and E is Young's modulus. In addition, the curvatures must satisfy the compatibility condition

$$d(rK_\theta)/dr = K_r = -d^2w/dr^2 \tag{4}$$

while the moments and shear force Q must satisfy the equilibrium equations

$$\begin{aligned} d(rM_r)/dr - M_\theta &= rQ \\ d(rQ)/dr &= -rp - \sum_i P_i\delta(r-r_i)r_i \end{aligned} \tag{5}$$

where δ is the Dirac operator.

The admissible class $\mathcal{S}$ of design variables $S(r)$ is defined as follows: (a) the condition of prescribed volume is met

$$V_0 = \int Sr\,dr. \tag{6}$$

(b) $S(r)$ is bounded and has, at most, a finite number of discontinuties

$$0 \le S(r) < \infty. \tag{7}$$

The design for minimum compliance entails minimizing W over all $S(r)$ lying is $\mathcal{S}$.

Definition 1. The design $S = S^*$ is locally a minimum (maximum) design whenever $S^* \in \mathcal{S}$ and there exists a number $\epsilon > 0$ such that $W(S^*) \le (\ge) W(S)$ for all neighboring designs $S \in \mathcal{S}$ satisfying $\max|S - S^*| \le \epsilon$.

Definition 2. The design $S = S^{**}$ is an absolute minimum design whenever $W(S^{**}) \le W(S)$ for all other designs $S \in \mathcal{S}$.

Static Formulation. On account of the principle of virtual work, (1) may be expressed

$$J(S; M_r, M_\theta) = W/2\pi = \int S^{-3}(M_r^2 - 2\nu M_r M_\theta + M_\theta^2)r\,dr \tag{8}$$

Now, let $\mathcal{M}$ be the class of statically admissible bending moments (M_r, M_θ) where rM_r is continuous and meets the stress boundary conditions but M_θ is merely bounded. The static formulation entails minimizing J over all $S \in \mathcal{S}$ and all $(M_r, M_\theta) \in \mathcal{M}$. In this case, the definitions for local and absolute minima are the same as Def. 1 and 2 with W replaced by J. The equivalence between the two formulations is expressed in the following theorem.

Theorem 1†: Any admissible design S^* which is local (absolute) minimum for $W(J)$ is also a local (absolute) minimum for $J(W)$.

In view of the stated objective of finding local and absolute minima for W, it suffices to consider the statical formulation.

3. NECESSARY CONDITIONS FOR OPTIMALITY

Let the design S^* and bending moments (M_r^*, M_θ^*) provide a local minimum of J. The first necessary condition for optimality is obtained by taking admissible infinitesimal variations δS.‡ Introduction of the constant Lagrangean multiplier $-3k^{-4}$ to (6) together with the vanishing of the first variation furnishes

$$\int [k^{-4} - S^{*-4}(M_r^{*2} - 2\nu M_r^* M_\theta^* + M_\theta^{*2})]r\delta S\,dr = 0$$

†The proof of *Theorem* 1 is found in the Appendix.

‡It is unnecessary to consider variations in the moments, since they are known to provide the exact moments (M_r^*, M_θ^*) corresponding to S^*.

for all δS. Consequently the design must satisfy

$$S^{*4} = k^4(M_r^{*2} - 2\nu M_r^* M_\theta^* + M_\theta^{*2}). \tag{9}$$

A second necessary condition for optimality is obtained from consideration of the following admissible variations

$$r\delta S = \begin{cases} 0 \\ \xi_1 \\ -\xi_1 \\ 0 \end{cases} \quad \delta M_\theta = \begin{cases} 0 \\ \xi_0 \\ -\xi_0 \\ 0 \end{cases} \quad r\delta M_r = \begin{cases} 0 & a \le r \le d-\delta \\ \xi_0(r-d+\delta) & d-\delta < r \le d \\ -\xi_0(r-d-\delta) & d < r < d+\delta \\ 0 & d+\delta \le r \le b \end{cases} \tag{10}$$

where a is the inner edge of the plate ($a = 0$ for the full plate), b is the outer edge, d is an arbitrary radius where $S(d) \neq 0$, and ξ_0, ξ_1 and δ are infinitesimal quantities. The effects of variation (10) are depicted in Fig. 1. In view of (9) and the particular variation (10), the change in compliance δJ becomes

$$\delta J = 2\int_{d-\delta}^{d+\delta} [S^{*-3}(\delta M_\theta)^2 + 6S^{*-1}k^{-4}(\delta S)^2 - 6S^{*-4}(M_\theta^* - \nu M_r^*)\delta S\delta M_\theta] r\,\mathrm{d}r \tag{11}$$

where only lowest order quantities have been retained.

Since ξ_0 and ξ_1 are constants, let $\xi_0 = A\xi_1$. Evaluation of (11) furnishes

$$\delta J = 4[A^2 S^{*-3} - 6AS^{*-4}d^{-1}(M_\theta^* - \nu M_r^*) + 6S^{*-1}d^{-2}k^{-4}]\xi_1^2 \delta d \tag{12}$$

where S^*, M_θ^* and M_r^* are evaluated at $r = d$. Since $\delta J \ge 0$ for all values A, the discrimant of the quadratic expression in A must be non-positive, viz.

$$3(M_\theta^* - \nu M_r^*)^2 \le 2(M_r^{*2} - 2\nu M_r^* M_\theta^* + M_\theta^{*2}). \tag{13}$$

It is convenient to express (9) in the parametric form

$$\begin{aligned} M_r^* &= -S^{*2}k^{-2}\csc 2\alpha_0 \sin(\alpha - \alpha_0) \\ M_\theta^* &= S^{*2}k^{-2}\csc 2\alpha_0 \sin(\alpha + \alpha_0) \end{aligned} \quad 0 \le \alpha < 2\pi \tag{14}$$

$$\sin 2\alpha_0 = (1-\nu^2)^{1/2} \qquad \pi/4 \le \alpha_0 \le \pi/3.$$

In (14), the constant α_0 may be interpreted as a material parameter corresponding to the range

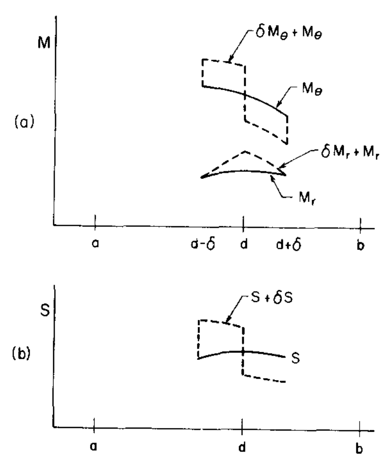

Fig. 1. Effect of variation (10). Stress variation (a) and design variation (b).

$0 \le \nu \le 1/2$. Substitution of (14) into (13) furnishes

$$|\cos(\alpha - \alpha_0)| \le \sqrt{(2/3)}. \tag{15}$$

The corresponding necessary conditions for J to be a local maximum, $|\cos(\alpha - \alpha_0)| \ge \sqrt{(2/3)}$, may be similarly derived from (12). The results of this section are summarized in *Theorem* 2.

Theorem 2: Given a design S^* with corresponding bending moments (M_r^*, M_θ^*). Then: (a) necessary conditions for J to be a local minimum are that (14) and (15) are satisfied everywhere that S does not vanish; (b) necessary conditions for J to be a local maximum are that (14) and $|\cos(\alpha - \alpha_0)| \ge \sqrt{(2/3)}$ are satisfied everywhere that S does not vanish; (c) sufficient conditions for J to be a saddle point are that (14) is satisfied and that $|\cos(\alpha - \alpha_0)| < \sqrt{(2/3)}$ over part of the plate and $|\cos(\alpha - \alpha_0)| > \sqrt{(2/3)}$ over another part of the plate.

4. CONTINUITY CONDITIONS

Let the design S^* with corresponding bending moments (M_r^*, M_θ^*) be a local minimum for the compliance J. Since $(M_r^*, M_\theta^*) \in \mathcal{M}$ and $S^* \in \mathcal{S}$, it follows that M_r^* is continuous everywhere. Moreover, continuity of K_θ at locations of vanishing stiffness may be established by considering variation (10) with ξ_0, ξ_1 and δ infinitesimal. Routine calculations furnish $\delta J = 2[K_\theta(d^-) - K_\theta(d^+)]\xi_0\delta d$. The result follows by noting that the sign of ξ_0 is arbitrary. An alternative approach is presented by Masur[4]. If S is discontinuous, then (3) shows that K_r is also discontinuous while (9) requires a discontinuity in M_θ^*.

Let a discontinuity in S^* occur at $r = c$. Denote the left and right-hand limits of S^* at c by S_-^* and S_+^*, respectively. Also choose two radii c_1, c_2, where $c_1 < c < c_2$ such that c is the only discontinuity in $c_1 \le r \le c_2$. Now consider the following variation†

$$r\delta S = \begin{cases} 0 \\ \eta_0 \\ -\eta_0 \\ 0 \end{cases} \quad \delta M_\theta = \begin{cases} 0 \\ \xi_0 \\ -\xi_1 \\ 0 \end{cases} \quad r\delta M_r = \begin{cases} 0 & r < c_1 \\ \xi_0(r - c_1) & c_1 \le r \le c \\ -\xi_1(r - c - \epsilon) & c < r \le c + \epsilon \\ 0 & c + \epsilon < r < c_2 \end{cases}. \tag{16}$$

The constants η_0, η_1, ξ_0 and ξ_1 are chosen so that both δS and $(\delta M_r, \delta M_\theta)$ are admissible and so that the variations affect an infinitesimal change in the discontinuity circle from c to $c + \epsilon$ (Fig. 2). Thus

$$\begin{aligned} \eta_1\epsilon &= \eta_0(c - c_1) = (S_+^* - S_-^*)c\epsilon \\ \xi_1\epsilon &= \xi_0(c - c_1) = \epsilon(M_{\theta+} - M_{\theta-}^*). \end{aligned} \tag{17}$$

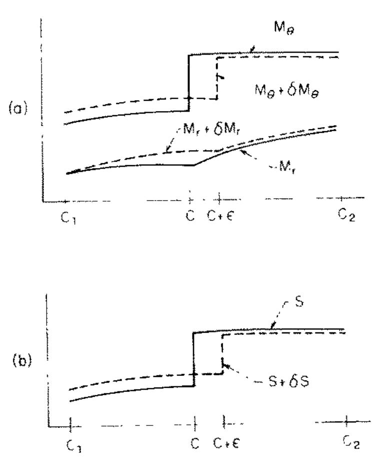

Fig. 2. Discontinuity variation (16). Stress variation (a) and design variation (b).

†This particular variation was first introduced by Megarefs[5].

In the region $c_1 \leq r \leq c$, the change in compliance δJ_1 becomes

$$\delta J_1 = -\frac{3}{k^4}\int_{c_1}^{c} \delta S r \, dr + 2\int_{c_1}^{c} (K_r^* \delta M_r + K_\theta^* \delta M_\theta) r \, dr$$
$$= \left[-\frac{3}{k^4}(S_+^* - S_-^*) + 2K_\theta^*(M_{\theta+}^* - M_{\theta-}^*)\right] c\epsilon. \tag{18}$$

In the region $c \leq r \leq c + \epsilon$, the change in compliance δJ_2 is

$$\delta J_2 = \int_{c}^{c+\epsilon} [F(M_r^* + \delta M_r, M_\theta^* + \delta M_\theta, S^* + \delta S) - F(M_r^*, M_\theta^*, S^*)] r \, dr \tag{19}$$

where

$$F = (M_r^2 - 2\nu M_\theta M_r + M_\theta^2) S^{-3}.$$

Recalling the effect of (16), the first term in the integrand of (19) is, to within terms of order ϵ, $F(c^-)$ which by virtue of (9) becomes S_-^*/k^4. Similarly the second term becomes S_+^*/k^4, and therefore

$$\delta J_2 = (S_- - S_+) c\epsilon / k^4. \tag{20}$$

Since S^* is an assumed minimum, $\delta J = \delta J_1 + \delta J_2 \geq 0$. On the other hand, a variation similar to (16) which moves the discontinuity circle from c to $c - \epsilon$ will produce compliance changes identical to (18) and (20) except for sign. Consequently, it is necessary that $\delta J = 0$, i.e.

$$[2S^*/k^4 - M_\theta^* K_\theta^*]|_{c^-}^{c^+} = 0$$

or equivalently,

$$S^*(3 \sin 2\alpha - \sin 2\alpha_0)|_{\alpha^-}^{\alpha^+} = 0. \tag{21}$$

Equation (17) shows that although ξ_0 and η_0 are infinitesimal, both ξ_1 and η_1 are not. Since S^* is not neighboring the design $S^* + \delta S$ everywhere, it follows that S^* may be a local minimum even though (21) does not hold. What, however, has been established may be summarized as follows:

Theorem 3: Continuity of $S^*(3 \sin 2\alpha - \sin 2\alpha_0)$ is a necessary condition that S^* with corresponding bending moments (M_r^*, M_θ^*) is the smallest (largest) local minimum (maximum) for J.

Equation (21), together with continuity of K_θ^* and M_r^* may be used to establish

Corollary 1:† If S^* is the smallest local minimum of J, then S^* is continuous everywhere. Discontinuities in α are permitted provided both

$$\cos(\alpha^- - \alpha_0) = \cos(\alpha^+ - \alpha_0) = 0 \tag{22}$$

and S^* vanishes at the circle of discontinuity. At such circles, K_θ is bounded and continuous although K_r is infinite. Since (22) satisfies (15), the largest local maximum cannot have a discontinuity.

5. EXAMPLE

With the aid of (6) and (14), equations (3) and (8) become

$$K_r = S^{-1}k^{-2}\cos(\alpha + \alpha_0) \qquad K_\theta = S^{-1}k^{-2}\cos(\alpha - \alpha_0) \tag{23}$$
$$J = k^{-4} V_0. \tag{24}$$

†The proof is algebraic and therefore omitted. For sandwich plates the corresponding theorem and corollary were immediately duducible from the continuity of M_r^* and K_θ^*[6].

Now consider the full plate with radius b and edge moments M_0 applied to the outer edge. Assume an isotropic state of stress which automatically satisfies (15). Equations (5), together with the edge condition, furnish

$$M_r = M_\theta = M_0 \qquad \alpha = 0$$

from which (9) furnishes

$$S = k(2\cos\alpha_0 M_0)^{1/2}.$$

Substituting S into (6) yields

$$k = \sqrt{(2)}\,V_0/b^2\sqrt{(M_0\cos\alpha_0)}$$

so that (24) provides

$$J = b^8 M_0^2\, cos^2\,\alpha_0/4V_0^3. \tag{25}$$

The curvatures (23) become

$$K_r = K_\theta = b^6 M_0 \cos\alpha_0/4V_0^3$$

which satisfies compatibility (4). Consequently, the uniform design

$$S^* = 2V_0/b^2 \tag{26}$$

satisfies all the necessary criterion for a local minimum.

Now consider the alternative design obtained by assuming that K_r vanishes identically. Thus

$$\alpha = \pi/2 - \alpha_0. \tag{27}$$

Substitution of (27) into (14) together with eqn (23) furnishes

$$K_\theta = S^{-1}k^{-2}\sin 2\alpha_0 \tag{28}$$

$$\begin{aligned} M_r &= -S^2k^{-2}\cot 2\alpha_0 \\ M_\theta &= S^2k^{-2}\csc 2\alpha_0. \end{aligned} \tag{29}$$

Substitution of (28) into (4), and (29) into (5) furnishes, respectively,

$$\begin{aligned} S &= c_0 r \\ \cos 2\alpha_0 &= -\nu = -1/3. \end{aligned} \tag{30}$$

The constants c_0 and k are obtained from (6) and the edge condition. Therefore

$$c_0 = 3V_0/b^3 \qquad k^2 = -9V_0^2\cot 2\alpha_0/M_0 b^4. \tag{31}$$

Since $\nu = 1/3$, it follows that $\alpha_0 = 54{\cdot}74°$ and $\alpha = 35{\cdot}26°$. Finally, (24) furnishes the compliance

$$J = 8M_0^2 b^8/81V_0^3. \tag{31}$$

It may easily be shown that $\cos^2(\alpha - \alpha_0) > 2/3$ and consequently (30) satisfies the necessary criterion for a local maximum. With $\nu = 1/3$, the quotient of (31) and (25) becomes

$$\frac{J_{max}}{J_{min}} = \frac{32}{27}. \tag{32}$$

6. THE ABSOLUTE MINIMUM

In the preceding sections necessary criterion for the determination of the local minimum and smallest local minimum were developed. It will now be shown that the compliance may be made arbitrarily small.

Consider the full plate simply supported at its radius b. Suppose further that the loading is unidirectional and does not contain a concentrated force at the center of the plate. Under these conditions the shear will be non-positive and bounded. Let n be a large number and subdivide the radius b into n equal increments of length δ.

$$n\delta = b. \tag{33}$$

Define a quantity ϵ by

$$n^2\epsilon = b \tag{34}$$

so that $\epsilon \ll \delta$.

Now choose the admissible stresses (Fig. 3b)

$$\begin{aligned} &M_r = M_\theta = \int_{(i+1)\delta}^{r} Q(\xi)\,\mathrm{d}\xi \qquad i\delta + \epsilon_i \le r \le (i+1)\delta \\ &\left.\begin{aligned} &M_\theta = \xi_i \\ &rM_r = \xi_i(r - i\delta) + \int_{i\delta}^{r} \xi Q(\xi)\,\mathrm{d}\xi \end{aligned}\right\} \quad i\delta < r < i\delta + \epsilon_i \end{aligned} \tag{35}$$

where $(i = 0, 1, \ldots, n-1)$, $\xi_0 = \epsilon_0 = 0$ and $\epsilon_i = \epsilon$ for $i \ge 1$. Continuity of rM_r at $r = i\delta + \epsilon$ $(i \ge 1)$ requires

$$rM_r|_{i\delta+\epsilon} = \xi_i\epsilon = -ib^2n^{-2}Q|_{(i+1)\delta} \tag{36}$$

where terms of $\mathcal{O}(n^{-3})$ are omitted.

With the admissible stiffness (Fig. 3a)

$$S^3 = \begin{cases} M_r & i\delta + \epsilon_i \le r \le (i+1)\delta \\ \eta^3 = \text{constant} & i\delta < r < i\delta + \epsilon_i \end{cases} \tag{37}$$

the compliance J becomes

$$J = 2(1-\nu)\sum_{i=0}^{n-1}\int_{i\delta+\epsilon_i}^{(i+1)\delta} M_r r\,\mathrm{d}r + \sum_{i=1}^{n-1}\int_{i\delta}^{i\delta+\epsilon_i}\frac{\xi_i^2}{\eta^3} r\,\mathrm{d}r. \tag{38}$$

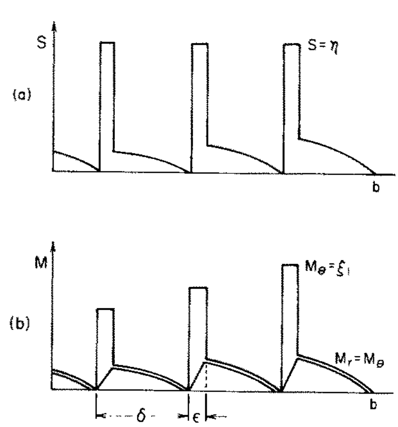

Fig. 3. Design for arbitrarily small compliance (a) and corresponding bending moments (b).

The constant η is not arbitrary; it must be chosen so that (6) is satisfied,

$$V_0 = \sum_{i=0}^{n-1} \int_{i\delta}^{i\delta+\epsilon_i} \eta r \, dr + \sum_{i=0}^{n-1} \int_{i\delta+\epsilon_i}^{(i+1)\delta} (M_r)^{1/3} r \, dr. \tag{39}$$

With (33) and (34), equation (39) becomes

$$V_0 = \frac{\eta b^2}{2n} + \mathcal{O}(n^{-4/3})$$

so that

$$\eta = 2V_0 b^{-2} n + \mathcal{O}(n^{-1/3}). \tag{40}$$

Finally, (33), (34) and (40) will reduce (38) to

$$J = \mathcal{O}(n^{-1})$$

which approaches zero as $n \to \infty$. Note that the limit design S

$$S = \lim_{n\to\infty} S(r, n)$$

becomes unbounded on every closed subinterval of $[0, b]$, and consequently S is not contained in $\mathcal{S}$.

7. ASSOCIATED PLASTIC PLATE

In a recent paper[6] the complete equivalence between the elastic design for minimum compliance and plastic design for minimum weight was established for sandwich structures whose face sheets obey the yield condition

$$M_0^2 = M_r^2 - 2\nu M_r M_\theta + M_\theta^2 \tag{41}$$

where ν has the same value as Poisson's ratio of the corresponding elastic plate and M_0 is the yield moment. The relationship between these two designs will now be examined for solid plates.

Consider a rigid-perfectly plastic plate identical to the elastic plate of the previous sections. The loading and support conditions are the same as for the elastic plate. For the yield condition (41), it is desired to design the thickness for minimum volume while insuring safety against incipient plastic collapse. Define the plastic stiffness S_p by

$$S_p = M_0^{1/2} = S_p(M_r, M_\theta). \tag{42}$$

The problem may now be posed as follows: Minimize Φ

$$\Phi = \int_a^b S_p(M_r, M_\theta) r \, dr \tag{43}$$

over all $(M_r, M_\theta) \in \mathcal{M}$. This is a standard "calculus of variations" problem.†

The Euler (extremal) equation is

$$\frac{d}{dr}\left(r \frac{\partial S_p}{\partial M_\theta}\right) = \frac{\partial S_p}{\partial M_r}. \tag{44}$$

The gradient $(\partial S_p / \partial M_\theta, \partial S_p / \partial M_r)$ may be identified with the incipient curvature rates at

†This is formally the same as the standard functional $\int F(y, y', r)\, dr$. However, in the calculus of variations, y (i.e. M_r) is the design variable. A relative minimum is defined in terms of neighboring curves to y; thus $\delta y'(\delta M_\theta)$ is not required to be small. When S_p is the design variable, small variations δS_p imply, through (41), that δM_r and δM_θ are both also small. A true local optimum, therefore, is equivalent to the "weak" optimum in variational calculus[8].

collapse[7], and (44) becomes merely the compatibility condition (4). Thus the field equations for the extremals of the plastic design are formally identical to those for the extremals of the elastic design.

The second necessary condition for a local minimum (maximum) is the Legendre condition†

$$\frac{\partial^2 S_p}{\partial M_\theta^2} \geq (\leq) 0. \tag{45}$$

When (41) is expressed parametrically by (14) (with $k \equiv 1$), eqn (45) simplies to

$$|\cos(\alpha - \alpha_0)| \leq (\geq) \sqrt{(2/3)}$$

which is precisely the same necessary condition obtained above for elastic plates.

When discontinuities in M_θ and S_p are considered, the necessary continuity conditions of Wierstrass are applicable, namely

$$\left.\frac{\partial S_p}{\partial M_\theta}\right|_{c^-} = \left.\frac{\partial S_p}{\partial M_\theta}\right|_{c^+} \tag{46}$$

$$S_p - M_\theta \left.\frac{\partial S_p}{\partial M_\theta}\right|_{c^-} = S_p - M_\theta \left.\frac{\partial S_p}{\partial M_\theta}\right|_{c^+}. \tag{47}$$

Equation (46) is equivalent to continuity in the curvature rate $\dot{K}_\theta$, while (47)‡ is precisely the same as for elastic design

$$S_p[3 \sin 2\alpha - \sin 2\alpha_0]|_{\alpha^-}^{\alpha^+} = 0.$$

Therefore, Theorem 3 and Corollary 1§ are also valid for the plastic plate.

It is known that the plastic plate does not possess an absolute minimum[10, 11]. For example, consider the admissible stresses of Fig. 3a. Then

$$J = \int_0^b rS_p \, dr = \sum_{i=0}^{n-1} \int_{i\delta+\epsilon_i}^{(i+1)\delta} r\sqrt{[2(1-\nu)]} M_r^{1/2} \, dr + \sum_{i=0}^{n-1} \int_{i\delta}^{i\delta+\epsilon_i} \xi_i^{1/2} r \, dr$$
$$= \mathcal{O}(n^{-1/2}).$$

Finally, the sufficiency conditions for a local minimal are satisfaction of (44), the strict inequality (45) and the Jacobi condition[8]. In view of the exceptionally strong established correlation between the elastic and plastic problems, it is highly probable that these conditions are also sufficient for the elastic design, although a proof is presently lacking.

8. CONCLUSION

It has been shown that the extremal condition (9) may lead to designs that are either local minima or local maxima. A further necessary condition for a local minimum is derived from the second variation and is given by (15). These results are completely compatible with the investigation of Masur[2], but differ with the publications by Huang[1] and Shield[13] where (9) is also presented as a sufficient condition.

Except for a strict proof of the sufficiency conditions for a local minimal compliance design, the elastic design problem has been shown to be equivalent to the plastic design problem for the

†Mróz[9] appears to be the first to use this condition in plastic design.

‡This condition is established by Bolza[8] by considering a variation in the standard problem for which δy is small but $\delta y'$ is large. Such variations are valid when y is the design variable but not when S_p is the design variable. Therefore such variations are permissible in plastic design only when searching for the smallest (largest) local minimum (maximum).

§The result that S_p must vanish at discontinuities appears to be new even though this condition has been used before[12, 13]. The investigators[12, 13] treated the full clamped plate obeying the Tresca condition and obtained a solution by assuming $M_r = M_\theta$ in the central region and $M_\theta = 0$, $M_r < 0$ in the outer region. At the juncture, continuity of M_r together with the yield condition resulted in $S_p = 0$ for their particular design.

yield criterion (41). This result is also at variance with Huang's concluding remarks[1]. Huang restricts the design variable S to be continuous and differentiable. The theory presented in this paper shows that the least local minimum design is continuous (although it was not restricted to be so) but is not differentiable at the jumps in α. Thus, the unnecessary restriction that S be differentiable would show that even a local minimum would not exist for many problems; the same would be true even for the simpler case of sandwich plates.

Due to the high non-linearity in the field equations for a local minimal design, the literature is noticeable lacking with solutions for specific plate examples. To the author's knowledge, Huang[1] worked the only published example ($\nu = 1/3$), although Freiberger and Tekinalp[14] solved the same problem in plastic design ($\nu = 1/2$) a number of years earlier. It may be shown that Huang's solution does indeed satisfy (15) and therefore satisfies the second necessary condition for a minimum.

For many problems local maxima will not exist. Thus a discontinuity in α and (22) preclude the possibility of a local maximum. Similarly, so will the isotropic condition ($\alpha = 0$) at the center of the plate. On the other hand, the condition $M_r = 0$, $S \neq 0$ at a free or supported edge of the plate will preclude a local minimum design.

Finally, the result that the compliance may be made arbitrarily close to zero is, of course, physically absurd. The designs corresponding to such compliances are admissible according to present terminology; however, they flagrantly violate the thin plate assumption. A more realistic approach would place a specific upper bound on S. An alternative approach might be to reformulate the problem based on a thick plate theory.

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APPENDIX

Proof of Theorem 1: Let S^* be a local minimum for W and let M_r^* and M_θ^* be the corresponding bending moments under the loads $p(r)$ and P_i. Let S be any design in $\mathcal{S}$ which is neighboring S^*, and (M_r, M_θ) any bending moments in $\mathcal{M}$. [*Note* (M_r, M_θ) is not necessarily neighboring (M_r^*, M_θ^*).] Then, according to the principle of minimum complimentary energy

$$W(S) \leq 2\pi J(S; M_r, M_\theta) \tag{48}$$

while Def. 1 and (8) furnishes

$$2\pi J(S^*; M_r^*, M_\theta^*) = W(S^*) \leq W(S). \tag{49}$$

Comparison of (48) with (49) furnishes

$$J(S^*; M_r^*, M_\theta^*) \leq J(S; M_r, M_\theta). \tag{50}$$

Next, it must be established that local minima of J are also local minima of W. Let S^* now be the local optimum of J, and (M_r^*, M_θ^*) be the actual bending moments at optimality. Thus (50) must be satisfied for all S neighboring S^* in $\mathcal{S}$ and all (M_r, M_θ) lying in $\mathcal{M}$. Now choose (M_r, M_θ) to be those moments that are actually produced by the given loading on the design S, so that the right-hand side of (50) becomes $W(S)/2\pi$. Equation (50) becomes

$$W(S^*)/2\pi = J(S^*; M_r^*, M_\theta^*) \leq W(s)/2\pi$$

completing the proof of the theorem.